

Duality Theorems for Infinite Braided Hopf Algebras

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Abstract

Let H be an infinite-dimensional braided Hopf algebra and assume that the braiding is symmetric on H and its quasi-dual H^d . We prove the Blattner-Montgomery duality theorem, namely we prove

$$(R\#H)\#H^d \cong R \otimes (H\#H^d) \quad \text{as algebras in braided tensor category } \mathcal{C}.$$

In particular, we present two duality theorems for infinite braided Hopf algebras in the Yetter-Drinfeld module category.

Keywords: braided Hopf algebra, duality theorem.

0 Introduction

The duality theorems play an important role in actions of Hopf algebras (see [10]). In [4] and [10], Blattner and Montgomery proved the following duality theorem for an ordinary Hopf algebra H and some Hopf subalgebra U of H° :

$$(R\#H)\#U \cong R \otimes (H\#U) \quad \text{as algebras,}$$

where R is a U -comodule algebra. The dual theorems for co-Frobenius Hopf algebra H ,

$$(R\#H)\#H^{*rat} \cong M_H^f(R) \quad \text{and} \quad (R\#H^{*rat})\#H \cong M_H^f(R) \quad \text{as } k\text{-algebras}$$

were proved in [5] (see [3, Corollary 6.5.6 and Theorem 6.5.11]). On the other hand, braided Hopf algebras have attracted much attention in both mathematics and mathematical physics (see [1][6] [9]). One of the authors in [14] generalized the duality theorem to the braided case, i.e., for a finite Hopf algebra H with $C_{H,H} = C_{H,H}^{-1}$,

$$(R\#H)\#H^* \cong R \otimes (H\bar{\otimes} H^*) \quad \text{as algebras in } \mathcal{C}.$$

The Blattner-Montgomery duality theorem was also generalized to Hopf algebras over commutative rings [2].

In this paper we generalize the above results to infinite braided Hopf algebras. A braided Hopf algebra is called an infinite braided Hopf algebra if it has no left duals (See [12]). An important example of infinite braided Hopf algebras is the universal enveloping algebra of a Lie superalgebra. So the duality theorems for infinite braided Hopf algebras should have important applications in both mathematics and mathematical physics.

This paper is organized as follows. In section 1, since it is possible that $\text{Hom}(H, I)$ is not an object in \mathcal{C} for braided Hopf algebra H , we introduce quasi-dual H^d of H , and prove the duality theorem in a braided tensor category \mathcal{C} , i.e. $(R \# H) \# H^d \cong R \otimes (H \# H^d)$ as algebras in \mathcal{C} . In section 2, we concentrate on the Yetter-Drinfeld module category ${}^B_B\mathcal{YD}$, and show

$$(R \# H) \# U \cong R \otimes (H \# U) \quad \text{and} \quad (R \# U) \# V \cong R \otimes (U \# V)$$

as algebras in ${}^B_B\mathcal{YD}$. Here U and V are certain braided Hopf subalgebras of H° and H , respectively.

1 Duality theorem for braided Hopf algebras

In this section, we obtain the duality theorem for braided Hopf algebras living in the braided tensor category \mathcal{C} .

Let $(\mathcal{C}, \otimes, I, C)$ be a braided tensor category, where I is the identity object and C is the braiding. we write $W \otimes f$ for $\text{id}_W \otimes f$ and $f \otimes W$ for $f \otimes \text{id}_W$. The proofs in this section are very similar to those for the corresponding results in [10, Chapter 9] and [15, Chapter 7], so we only give the sketch to the proofs. In particular, there are the proofs in [15] by using braiding diagrams.

Definition 1.1 *Let $(H, m, \eta, \Delta, \epsilon)$ in \mathcal{C} be a braided Hopf algebra. If there is a braided Hopf algebra H^d and a morphism \langle, \rangle from $H^d \otimes H$ to I such that*

$$\begin{aligned} (\langle, \rangle \otimes \langle, \rangle)(H^d \otimes C \otimes H)(\Delta_{H^d} \otimes H \otimes H) &= \langle, \rangle (H^d \otimes m_H), \\ \epsilon_{H^d} &= \langle, \rangle (H^d \otimes \eta_H), \\ \langle, \rangle (m_{H^d} \otimes H) &= (\langle, \rangle \otimes \langle, \rangle)(H^d \otimes C \otimes H)(H \otimes H \otimes \Delta), \\ \langle, \rangle (\eta_{H^d} \otimes H) &= \epsilon_H, \\ \langle, \rangle (S_{H^d} \otimes H) &= \langle, \rangle (H^d \otimes S_H), \end{aligned}$$

then H^d is called a quasi-dual of H under \langle, \rangle .

Lemma 1.2 Let H^d be a quasi-dual of H under $<, >$ and $C_{H,H} = C_{H,H}^{-1}$. Assume that $(H \otimes <, >)(C_{H^d,H} \otimes H) = (H \otimes <, >)(C_{H,H^d}^{-1} \otimes H)$ implies $C_{H^d,H} = C_{H,H^d}^{-1}$, and $(H \otimes <, >)(C_{H^d,H^d} \otimes H) = (H^d \otimes <, >)(C_{H^d,H^d}^{-1} \otimes H)$ implies $C_{H^d,H^d} = C_{H^d,H^d}^{-1}$. Then $C_{U,V} = (C_{V,U})^{-1}$, for $U, V = H$ or H^d .

If $C_{H,H} = C_{H,H}^{-1}$, then we say that the braiding is symmetric on H . If $C_{U,V} = C_{V,U}^{-1}$ for $U, V = H$ or H^d , then we say that the braiding is symmetric on H and H^d . Throughout this section we assume that the braiding is symmetric on H and H^d .

Lemma 1.3 (i) (H^d, \rightharpoonup) is a left H -module algebra under the module operation. $\rightharpoonup = (H^d \otimes <, >)(H^d \otimes C)(C \otimes H^d)(H \otimes \Delta)$.

(ii) (H, \rightharpoonup) is a left H^d -module algebra under the module operation $\rightharpoonup = (H \otimes <, >)(C \otimes H)(H^d \otimes \Delta)$.

Consequently, we can construct two smssh products $H \# H^d$ and $H^d \# H$.

Definition 1.4 Assume the category \mathcal{C} is a subcategory of category \mathcal{D} . We say that CRL-condition holds on H and H^d under $<, >$ if the following conditions are satisfied:

(i) $E =: \text{End}_{\mathcal{D}} H$ is an algebra under multiplication of composition in \mathcal{C} and there exists a morphism $\text{val} : E \otimes H \rightarrow H$ in \mathcal{D} such that $\text{val}(f \otimes H) = \text{val}(g \otimes H)$ implies $f = g$ for any two morphisms f and g in \mathcal{C} from U to E , where U is an object in \mathcal{C} .

(ii) There are two morphisms $\rho : H^d \# H \rightarrow E$ and $\lambda : H \# H^d \rightarrow E$ in \mathcal{D} such that $\text{val}(\lambda \otimes H) = (m \otimes <, >)(H \otimes C \otimes H)(H \otimes H^d \otimes \Delta)$ and $\text{val}(\rho \otimes H) = m(<, > \otimes C)(H^d \otimes C \otimes H)(H \otimes H^d \otimes \Delta)$.

(iii) $\text{Im}(\lambda)$ is an object in \mathcal{D} and there exists a morphism $\bar{\lambda}$ in \mathcal{D} from $\text{Im}(\lambda)$ to $H \# H^d$ such that $\bar{\lambda}\lambda = \text{id}_{H \# H^d}$.

(iv) $C_{E,V}(\lambda \otimes V) = (V \otimes \lambda)C_{H \# H^d, V}$ and $C_{E,V}(\rho \otimes V) = (V \otimes \rho)C_{H^d \# H, V}$ for any object V in \mathcal{C} .

Lemma 1.5 λ is an algebra morphism from $H \# H^d$ to E and ρ is an anti-algebra morphism from $H^d \# H$ to E .

Proof. We only need show that $\text{val}((\lambda \otimes H)(m \otimes H)) = \text{val}((m \otimes H)(\lambda \otimes \lambda \otimes H))$ and $\text{val}((\rho \otimes H)(m \otimes H)) = \text{val}((m \otimes H)(\rho \otimes \rho \otimes H)(C \otimes H))$. The proof is similar to that of [10, Lemma 9.4.2]. \square

Lemma 1.6 The following relation holds: $m(\lambda \otimes \rho) = m(\rho \otimes \lambda)(H^d \otimes \rightharpoonup \otimes \leftharpoonup \otimes H^d)(H^d \otimes C_{H,H^d} \otimes C_{H^d,H} \otimes H^d)(H^d \otimes H \otimes C_{H^d,H^d} \otimes H \otimes H^d)(H^d \otimes C_{H^d,H} \otimes C_{H,H^d} \otimes H^d)(C_{H^d,H^d} \otimes H \otimes H \otimes C_{H^d,H^d})(S \otimes H^d \otimes H \otimes H \otimes H^d \otimes H^d)(\Delta \otimes H \otimes H \otimes \Delta)(H^d \otimes C_{H,H} \otimes H)(C_{H,H^d} \otimes C_{H^d,H})(H \otimes C_{H^d,H^d} \otimes H)$.

Proof. We show the relation after the following five steps. First we check that the relation holds on $(H \otimes \eta_{H^d}) \otimes (\eta_{H^d} \otimes H)$, $(\eta_H \otimes H^d) \otimes (H^d \otimes \eta_H)$, $(H \otimes \eta_{H^d}) \otimes (H^d \otimes \eta_H)$ and $(\eta_H \otimes H^d) \otimes (\eta_{H^d} \otimes H)$, respectively. Using these we check that the relation holds on $(H \otimes H^d) \otimes (H^d \otimes H)$. \square

Lemma 1.7 $R \# H$ becomes an H^d -module algebra under the module operation $\rightarrow' = (R \otimes \rightarrow)(C_{H^d, R} \otimes H)$.

Proof. It is straightforward. \square

Consequently, we obtain another smash product $(R \# H) \# H^d$.

If (R, ψ) is a right H^d -comodule algebra, then (R, α) becomes a left H -module algebra (see [9, Lemma 1.6.4]) under the module operation: $\alpha = (R \otimes \langle, \rangle)(R \otimes C_{H, H^d})(C_{H, R} \otimes R)(H \otimes \psi)$.

Theorem 1.8 Let H be a Hopf algebra. Assume that the CRL-condition holds on H and H^d under \langle, \rangle , and both H and H^d have invertible antipodes. Let R be an H^d -comodule algebra, so that R is an H -module algebra defined as above. Let H^d act on $R \# H$ by acting trivially on R and via \rightarrow on H , then

$$(R \# H) \# H^d \cong R \otimes (H \# H^d) \quad \text{as algebras in } \mathcal{D}.$$

In addition, if $\bar{\lambda}\rho(id_{H^d} \otimes \eta_H)$ is a morphism in \mathcal{C} from H^d to $H \# H^d$, then the above isomorphism is one in \mathcal{C} .

Proof. By (CRL)-condition, there exists a morphism $\bar{\lambda}$ in \mathcal{D} from $Im(\lambda)$ to $H \# H^d$ such that $\bar{\lambda}\lambda = id_{H \# H^d}$. We first define a morphism $w = \bar{\lambda}\rho(S^{-1} \otimes \eta_H)$ from H^d to $H \# H^d$. Since ρ and S^{-1} are anti-algebra morphisms by Lemma 1.5, w is an algebra morphism.

We now define two morphisms $\Phi = (R \otimes m_{H \# H^d})(R \otimes w \otimes H \otimes H^d)(\psi \otimes H \otimes H^d)$ from $(R \# H) \# H^d$ to $R \otimes (H \# H^d)$ and $\Psi = (R \otimes m_{H \# H^d})(R \otimes w \otimes H \otimes H^d)(R \otimes S \otimes H \otimes H^d)(\psi \otimes H \otimes H^d)$ from $R \otimes (H \# H^d)$ to $(R \# H) \# H^d$. It is straightforward to verify that $\Phi\Psi = id$ and $\Psi\Phi = id$. To see that Φ is an algebra morphism, we only need to show that $\Phi' = (R \otimes \lambda)\Phi$ is an algebra morphism. Set $\xi = (R \otimes \rho)(R \otimes S^{-1} \otimes \eta_H)\psi$ from R to $R \otimes (H \# H^d)$. We have that ξ is an algebra morphism and $\Phi' = (R \otimes m)(\xi \otimes \lambda)$. Using Lemma 1.6, we can show

$$\begin{aligned} (R \otimes m)(C \otimes E)(\lambda \otimes \xi) = \\ (R \otimes m)(\xi \otimes \lambda)(\alpha \otimes H \otimes H^d)(H \otimes C_{H, R} \otimes H^d)(\Delta \otimes C_{H^d, R}) \quad (*). \end{aligned}$$

We now show that Φ' is an algebra morphism. Indeed,

$$m_{R \otimes E}(\Phi' \otimes \Phi')$$

$$\begin{aligned}
&= (m \otimes m)(R \otimes C_{E,R} \otimes E)(m \otimes m)(\xi \otimes \lambda \otimes \xi \otimes \lambda) \\
&= (R \otimes m)(m \otimes E \otimes m)(R \otimes C_{E,R} \otimes m \otimes E)(R \otimes E \otimes C_{E,R} \otimes E \otimes E) \\
&\quad (\xi \otimes \lambda \otimes \xi \otimes \lambda)(R \otimes m)(m \otimes E \otimes m)(R \otimes C_{E,R} \otimes m \otimes E) \\
&\quad (R \otimes E \otimes \xi \otimes \lambda \otimes E)(R \otimes E \otimes \alpha \otimes H \otimes H^d \otimes E) \\
&= (R \otimes E \otimes H \otimes C_{H,R} \otimes H^d \otimes E)(\xi \otimes \Delta \otimes C_{H^d,R} \otimes \lambda) \quad \text{by } (*) \\
&= (m \otimes m)(R \otimes C_{E,R} \otimes m)(R \otimes E \otimes \xi \otimes \lambda)(R \otimes E \otimes \alpha \otimes m_{H \# H^d}) \\
&\quad (R \otimes E \otimes H \otimes C_{H,R} \otimes H^d \otimes H \otimes H^d) \\
&\quad (\xi \otimes \Delta \otimes C_{H^d,R} \otimes H \otimes H^d) \quad (\text{by Lemma 1.5}) \\
&= (R \otimes m)(\xi \otimes \lambda)(m \otimes H \otimes H^d)(R \otimes \alpha \otimes m_{H \# H^d})(R \otimes H \otimes C_{H,R} \otimes H^d \otimes H \otimes H^d) \\
&\quad (R \otimes \Delta \otimes C_{H^d,R} \otimes H \otimes H^d) \quad (\text{since } \xi \text{ is algebraic}) \\
&= \Phi' m_{(R \# H) \# H^d}.
\end{aligned}$$

Thus Φ' is algebraic and Φ is also algebraic. If $\bar{\lambda}\rho(id_{H^d} \otimes \eta_H)$ is a morphism in \mathcal{C} , then Φ is an isomorphism in \mathcal{C} . \square

We obtain the following by Theorem 1.8.

Corollary 1.9 *Let H be a finite braided Hopf algebra with a left dual H^* . If the braiding is symmetric on H , then*

$$(R \# H) \# H^* \cong R \otimes (H \# H^*) \quad \text{as algebras in } \mathcal{C}.$$

This corollary reproduces the main result in [14].

2 Duality theorems in the Yetter-Drinfeld module category

In this section, we present the duality theorem for braided Hopf algebras in the Yetter-Drinfeld module category $({}^B_B\mathcal{YD}, C)$. Throughout this section, H is a braided Hopf algebra in $({}^B_B\mathcal{YD}, C)$ with finite-dimensional Hopf algebra B and H^d is a quasi-dual of H under a left faithful \langle, \rangle (i.e. $\langle x, H \rangle = 0$ implies $x = 0$) such that $\langle b \cdot f, x \rangle = \langle f, S(b) \cdot x \rangle$ and $\sum \langle f_{(0)}, x \rangle f_{(-1)} = \sum \langle f, x_{(0)} \rangle S^{-1}(x_{(-1)})$ for any $x \in H, b \in B, f \in H^d$. Let b_B denote the coevaluation of B and \langle, \rangle_{ev} the ordinary evaluation of any spaces.

Lemma 2.1 *(i) If (V, α_V, ϕ_V) and (W, α_W, ϕ_W) are two Yetter-Drinfeld modules over B , then $\text{Hom}_k(V, W)$ is a Yetter-Drinfeld module under the following module operation and comodule operation: $(b \cdot f)(x) = \sum b_2 \cdot f(S(b_1) \cdot x)$ and $\phi(f) = (S^{-1} \otimes \hat{\alpha})(b_B \otimes f)$,*

where $\hat{\alpha}$ is defined by $(b^* \cdot f)(x) = \langle b^*, x_{(-1)} S(f(x_{(0)}))_{(-1)} \rangle_{ev} (f(x_{(0)}))_{(0)}$ for any $x \in V, f \in \text{Hom}_k(V, W), b^* \in B^*$. In particular, if V is an object in ${}^B_B\mathcal{YD}$, then so is V^* .

(ii) If V is an object in $({}^B_B\mathcal{YD}, C)$, then V^* is object in $({}^B_B\mathcal{YD}, C)$ and the evaluation \langle, \rangle_{ev} is a morphism in $({}^B_B\mathcal{YD}, C)$.

(iii) If the braiding is symmetric on V , then it is symmetric on V and V^* .

Proof. (i) It is clear that $\sum f_{(-1)} f_{(0)}(x) = \sum (f(x_{(0)}))_{(-1)} S^{-1}(x_{(-1)}) \otimes (f(x_{(0)}))_{(0)}$ for any $x \in V, f \in \text{Hom}_k(V, W), b \in B$. Using this, we can show that $\text{Hom}_k(V, W)$ is a B -comodule. Similarly, we can show that $\text{Hom}_k(V, W)$ is a B -module. We now show that

$$\sum (b \cdot f)_{(-1)} \otimes (b \cdot f)_{(0)} = b_1 f_{(-1)} S(b_3) \otimes b_2 \cdot f_{(0)} \quad (*)$$

for any $f \in \text{Hom}_k(V, W), b \in B$. For any $x \in V$, see that

$$\begin{aligned} \sum (b \cdot f)_{(-1)} \otimes (b \cdot f)_{(0)}(x) &= \sum b_1 (f(S(b_4)) \cdot x_{(0)})_{(-1)} S(b_3) S^{-1}(x_{(-1)}) \\ &\quad \otimes b_2 \cdot (f(S(b_4)) \cdot x_{(0)})_{(0)} \quad \text{and} \\ b_1 f_{(-1)} S(b_3) \otimes (b_2 \cdot f_{(0)})(x) &= b_1 f_{(-1)} S(b_4) \otimes b_2 \cdot f_{(0)}((S(b_3) \cdot x)) \\ &= \sum b_1 (f(S(b_4) \cdot x_{(0)}))_{(-1)} S(b_3) S^{-1}(x_{(-1)}) b_5 S(b_6) \\ &\quad \otimes b_2 \cdot (f(S(b_4) \cdot x_{(0)}))_{(0)} \\ &= \sum b_1 (f(S(b_4)) \cdot x_{(0)})_{(-1)} S(b_3) S^{-1}(x_{(-1)}) \\ &\quad \otimes b_2 \cdot (f(S(b_4)) \cdot x_{(0)})_{(0)}. \end{aligned}$$

Thus $(*)$ holds and $\text{Hom}_k(V, W)$ is a Yetter-Drinfeld module.

(ii) By (i), V^* is a Yetter-Drinfeld B -module. Obviously, \langle, \rangle is a B -module homomorphism. In order to show that \langle, \rangle is a B -comodule homomorphism, it is enough to prove that $\sum h_{(-1)}^* h_{(0)} \langle h_{(0)}^*, h_{(0)} \rangle = 1_B \langle h^*, h \rangle$ for any $h^* \in V^*, h \in V$. Indeed, the left side $= \sum S^{-1}(h_{(-1)2}) h_{(-1)1} \langle h^*, h_{(0)} \rangle = 1_B \langle h^*, h \rangle$. This complete the proof.

(iii) It follows from Lemma 1.2. \square

Lemma 2.2 Let A be a braided algebra in $\mathcal{C} = ({}^B_B\mathcal{YD}, C)$ and $A_{\mathcal{C}}^{\circ} = \{f \in H^* \mid \text{Ker}(f) \text{ contains an ideal of finite codimension in } {}^B_B\mathcal{YD}\}$. Then $A_{\mathcal{C}}^{\circ}$ is a braided coalgebra in $({}^B_B\mathcal{YD}, C)$, called the finite dual of A in \mathcal{C} and written as A° in short. Moreover, if H is a braided Hopf algebra in \mathcal{C} , then $H_{\mathcal{C}}^{\circ}$ is a braided Hopf algebra in \mathcal{C} .

Proof. By Lemma 2.1, A^* is a B -module and B -comodule. First we show that A° is an object in ${}^B_B\mathcal{YD}$. For any $f \in A^{\circ}$, there exists an ideal I of A and I is a B -submodule and a B -subcomodule of A with finite codimension and $f(I) = 0$. Since $(b \cdot f)(x) = f(S(b) \cdot x) = 0$ for any $b \in B, x \in I$, we have $b \cdot f \in A^{\circ}$. Thus A° is a B -submodule of A^* . By Lemma 2.1, we can assume $\phi_{A^*}(f) = \sum_i u_i \otimes v_i$ with linear

independent u'_i 's. Since $\sum_i u_i v_i(x) = \sum f(x_{(0)}) S^{-1}(x_{(-1)}) = 0$ for any $x \in I$, we have that $v_i(x) = 0$ and $v_i(I) = 0$, which implies $v_i \in A^\circ$. thus A° is a B -subcomodule of A^* .

We next show that $A^\circ \otimes A^\circ = (A \otimes A)^\circ$ and $m^*(A^\circ) \subseteq A^\circ \otimes A^\circ$ by using the method similar to the proof in [3, Lemma 1.5.2]. To show that m^*, η^* are morphisms in ${}^B_B\mathcal{YD}$, we only need show that if f is a morphism from U to V in ${}^B_B\mathcal{YD}$, then f^* is a morphism from V^* to U^* in ${}^B_B\mathcal{YD}$. Indeed, for any $v^* \in V^*, u \in U, b \in B$, see that

$$\begin{aligned} ((b \cdot f^*(v^*))(u)) &= (f^*(v^*))(S(b) \cdot u) \\ &= v^*(f(S(b) \cdot u)) \\ &= (f^*(b \cdot v^*))(u). \end{aligned}$$

Thus $(b \cdot f^*)(v^*) = f^*(b \cdot v^*)$ and f^* is a B -module homomorphism. Similarly, we can show that f^* is a B -comodule homomorphism. Consequently, (A°, m^*, η^*) is a braided coalgebra in $({}^B_B\mathcal{YD}, C)$. Finally we can similarly complete the other. \square

Let λ' denote the k -linear map from $H \# H^d$ to $End_k H$ by sending $h \otimes h^d$ to $\lambda'(h \# h^d)$ with $\lambda'(h \# h^d)(x) = h \langle h^d, x \rangle$ for any $x \in H, h \in H, h^d \in H^d$. Obviously, λ' is an injective k -linear map, so we can view $H \# H^d$ as a subspace of $End_k H$. Now we define λ and ρ . For any $h, x \in H, f \in H^d$, $(\lambda(h \# f))(x) =: (m \otimes \langle, \rangle)(H \otimes C \otimes H)(H \otimes H^d \otimes \Delta)(h \otimes f \otimes x) = \sum \langle f, x_{2(0)} \rangle h(S^{-1}(x_{2(-1)} \cdot x_1))$ and $(\rho(f \# h))(x) =: (\langle, \rangle \otimes m)(H^d \otimes H \otimes C)(H^d \otimes C \otimes H)(H^d \otimes H \otimes \Delta)(h \otimes f \otimes x) = \sum \langle f, h_{(-1)1} \cdot x_1 \rangle (h_{(-1)2} \cdot x_2) h_{(0)}$.

Let \mathcal{D} denote the category of vector spaces and $\mathcal{C} = {}^B_B\mathcal{YD}$. Define $val(f \otimes x) = f(x)$ for any $f \in E, x \in H$. If $\rho(H^d \# 1) \subseteq \lambda(H \# H^d)$ then we say that RL -condition holds on H and H^d under \langle, \rangle .

Lemma 2.3 *Let H be a braided Hopf algebra in $({}^B_B\mathcal{YD}, C)$ with $C_{H,H} = C_{H,H}^{-1}$.*

(i) *If the antipode of H is invertible, then there exists k -linear map $\bar{\lambda}$ from $Im\lambda$ to $H \# H^d$ such that $\bar{\lambda}\lambda = id_{H \# H^d}$.*

(ii) *If H is quantum cocommutative, then RL -condition holds on H and H^d under \langle, \rangle .*

(iii) *$C_{E,V}(\lambda \otimes V) = (V \otimes \lambda)C_{H \# H^d, V}$ and $C_{E,V}(\rho \otimes V) = (V \otimes \rho)C_{H^d \# H, V}$ for any object V in \mathcal{C} .*

(iv) *$E = End_k H$ is an algebra in ${}^B_B\mathcal{YD}$.*

(v) *If B is a commutative and cocommutative finite-dimensional Hopf algebra and H has an invertible antipode, then $\bar{\lambda}\rho(id_{H^d} \otimes \eta_H)$ is a morphism in ${}^B_B\mathcal{YD}$.*

Proof. (i) We define a k -linear map $\bar{\lambda}$ from $Im\lambda$ to $H \# H^d$ as follows: $\bar{\lambda}(f)(x) = m(f \otimes H)C(S^{-1} \otimes H)\Delta(x)$ for any $f \in Im\lambda, x \in H$. We can show that $\bar{\lambda}\lambda = id_{H \# H^d}$. Indeed, for any $h \in H, h^d \in H^d, x \in H$, we have

$$\bar{\lambda}\lambda(h \# h^d)(x) = m(m \otimes \langle, \rangle \otimes H)(H \otimes C \otimes H \otimes H)(H \otimes H^d \otimes \Delta \otimes H)$$

$$\begin{aligned}
& (H \otimes H^d \otimes C)(H \otimes H^d \otimes S^{-1} \otimes H)(H \otimes H^d \otimes \Delta)(h \# h^d)(x) \\
&= \lambda'(h \# h^d)(x).
\end{aligned}$$

Thus $\bar{\lambda}\lambda = id_{H \# H^d}$.

(ii) It follows from the simple fact $\rho(f \# 1) = \lambda(1 \# f)$ for any $f \in H^d$.

(iii) We only show that $C_{E,V}(\lambda \otimes V) = (V \otimes \lambda)C_{H \# H^d, V}$. Indeed, for any $h, x \in H, v \in V$, see that

$$\begin{aligned}
& (V \otimes \text{val})(C_{E,V} \otimes H)(\lambda \otimes V \otimes H)(h \otimes f \otimes v \otimes x) \\
&= \langle f, x_{(0)2(0)} \rangle (h_{(-1)}S^{-1}(x_{(0)2(-1)})_1 x_{(0)1(-1)}S^{-1}(x_{(0)1(-1)}S(S^{-1}(x_{(0)2(-1)})))_3) \cdot v \\
&\quad \otimes h_{(0)}S^{-1}(x_{(0)2(-1)})_2 \cdot x_{(0)1(0)}) \\
&= \langle f, x_{2(0)} \rangle h_{(-1)}S^{-1}(x_{2(-1)4})x_{1(-1)2}\underline{x_{2(-1)2}S^{-1}(x_{2(-1)1})}S^{-1}(x_{1(-1)1}) \cdot v \\
&\quad \otimes h_{(0)}(x_{2(-1)3} \cdot x_{1(0)}) \\
&= \langle f, x_{2(0)} \rangle h_{(-1)}S^{-1}(x_{1(-1)2}) \cdot v \otimes h_{(0)}(S^{-1}(x_{2(-1)1}) \cdot x_1) \quad \text{and} \\
& (V \otimes \text{val})(V \otimes \lambda \otimes H)(C_{H \# H^d, V} \otimes H)(h \otimes f \otimes v \otimes x) \\
&= \langle f_{(0)}, x_{2(0)} \rangle h_{(-1)} \cdot (f_{(-1)} \cdot v) \otimes h_{(0)}(S^{-1}(x_{2(-1)}) \cdot x_1) \\
&= \langle f, x_{2(0)} \rangle h_{(-1)}S^{-1}(x_{1(-1)2}) \cdot v \otimes h_{(0)}(S^{-1}(x_{2(-1)1}) \cdot x_1).
\end{aligned}$$

Thus $C_{E,V}(\lambda \otimes V) = (V \otimes \lambda)C_{H \# H^d, V}$. \square

(iv) It is straightforward.

(v) Let μ denote $\bar{\lambda}\rho(id_{H^d} \otimes \eta_H)$. We only show that μ is a B -module homomorphism. For any $x \in H, f \in H^d, b \in B$, since B is commutative and cocommutative, we have $\sum(b \cdot x)_{(-1)} \otimes (b \cdot x)_{(0)} = \sum x_{(-1)} \otimes (b \cdot x_{(0)})$ and

$$\begin{aligned}
(\mu(b \cdot f))(x) &= \sum \langle f, S(b)x_{1(-1)1} \cdot x_2 \rangle (x_{1(-1)2} \cdot x_3)S(x_{1(0)}) \\
(b \cdot \mu(f))(x) &= \langle f, x_{1(-1)1}S(b_4) \cdot x_2 \rangle (b_1x_{1(-1)2}S(b_5) \cdot x_3)(b_2S(b_3) \cdot S(x_{1(0)})) \\
&= \sum \langle f, S(b)x_{1(-1)1} \cdot x_2 \rangle (x_{1(-1)2} \cdot x_3)S(x_{1(0)}).
\end{aligned}$$

Thus μ is B -module homomorphism. \square

Every B -module category $({}_B\mathcal{M}, C^R)$ determined by quasitriangulr Hopf algebra (B, R) is a full subcategory of the Yetter-Drinfeld module category $({}_B^B\mathcal{YD}, C)$. Indeed, for any B -module (V, α) , define $\phi(v) = \sum R_i^{(2)} \otimes R_i^{(1)} \cdot v$ for any $v \in V$, where $R = \sum_i R_i^{(1)} \otimes R_i^{(2)}$. It is easy to check that (V, α, ϕ) is a Yetter-Drinfeld B -module. Similarly, every B -comodule category $({}_B\mathcal{M}, C^r)$ determined by coquasitriangulr Hopf algebra (B, r) is a full subcategory of the Yetter-Drinfeld module category $({}_B^B\mathcal{YD}, C)$.

Theorem 2.4 *Let H be a braided Hopf algebra in $({}^B_B\mathcal{YD}, C)$ with finite-dimensional B and $C_{H,H} = C_{H,H}^{-1}$. Assume that RL -condition holds on H and H^d under left faithful $<, >$, and both H and H^d have invertible antipodes. Let R be an H^d -comodule algebra, so that R is an H -module algebra defined as above. Let H^d act on $R\#H$ by acting trivially on R and via \rightharpoonup on H . Then*

$$(R\#H)\#H^d \cong R \otimes (H\#H^d) \quad \text{as } k\text{-algebras.}$$

Moreover, if B is commutative and cocommutative, then the above isomorphism is one as algebras in ${}^B_B\mathcal{YD}$.

Proof. It follows from Lemma 2.1 and Lemma 2.3 that (CRL) -condition in Definition 1.4 is satisfied. Considering Theorem 1.8, we complete the proof. \square

Corollary 2.5 (*Duality Theorem*) *Let H be a braided Hopf algebra in $({}^B_B\mathcal{YD}, C)$ with finite-dimensional B and $C_{H,H} = C_{H,H}^{-1}$. Assume that U is a braided Hopf subalgebra of H° and RL -condition holds on H and U under evaluation $<, >_{ev}$, and H has invertible antipode. Let R be an U -comodule algebra, so that R is an H -module algebra defined as above. Let U act on $R\#H$ by acting trivially on R and via \rightharpoonup on H , then*

$$(R\#H)\#U \cong R \otimes (H\#U) \quad \text{as } k\text{-algebras.}$$

Moreover, if B is commutative and cocommutative, then the above isomorphism is one as algebras in ${}^B_B\mathcal{YD}$.

Proof. It is clear that U is a quasi-dual of H under evaluation $<, >_{ev}$, which is a left faithful. U has an invertible antipode since H has an invertible antipode. By Theorem 2.4, we complete the proof. \square

Corollary 2.6 (*Second Duality Theorem*) *Let H be a braided Hopf algebra in $({}^B_B\mathcal{YD}, C)$ with finite-dimensional B and $C_{H,H} = C_{H,H}^{-1}$. Let U and V be a braided Hopf subalgebra of H° and H with invertible antipodes, respectively. Assume that RL -condition holds on U and V under $<, > = <, >_{ev} C$, and U is dense in H^* . Let R be an V -comodule algebra, so that R is an U -module algebra defined as above. Let V act on $R\#U$ by acting trivially on R and via \rightharpoonup on U , then*

$$(R\#U)\#V \cong R \otimes (U\#V) \quad \text{as } k\text{-algebras.}$$

Moreover, if B is commutative and cocommutative, then the above isomorphism is one as algebras in ${}^B_B\mathcal{YD}$.

Proof. It is clear that V is a quasi-dual of U under $\langle, \rangle = \langle, \rangle_{ev} C_{V,U}$ and \langle, \rangle is left faithful since U is dense. By Theorem 2.4, we can complete the proof. \square

Remark: In Corollary 2.5 H can be replaced by Hopf subalgebra V of H when $U \subseteq V^*$ and V has an invertible antipode.

Example 2.7 Let H be a quantum cocommutative braided Hopf algebra in $\mathcal{C} = {}^B_B\mathcal{YD}$ with finite-dimensional commutative and cocommutative B and $C_{H,H} = C_{H,H}^{-1}$ (for example, H is the universal enveloping algebra of a Lie superalgebra). Set $U = H_{\mathcal{C}}^{\circ} = A$. It is clear that (A, ϕ) is a right U -comodule algebra with $\phi = \Delta$. By Lemma 2.3, The RL -condition holds on H and U under evaluation \langle, \rangle_{ev} . By Corollary 2.5, we have

$$(R \# U) \# H \cong R \otimes (U \# H) \quad \text{as algebras in } {}^B_B\mathcal{YD}.$$

Remark: Although we have an efficient Sweedler's method (see [11]) to present co-operations, we suggest that readers use braiding diagrams to check all of our proofs because they are clearer.

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